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Social Network Analysis, Markov Chains and Input-output Models: Combining Tools to Map and Measure the Circulation of Currency in Small Economies

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Abstract

Localization is posited as the antidote for globalization, but little exists in the way of quantifying the micro-scale of small communities. In this paper, the reader's attention is drawn towards understanding that social network analysis, Markov chains and input-output models are equivalent, and that together these tools can be used to map and measure the circulation of currency in a small community. The "map" of the economy can be created using social network analysis, in a form equivalent to Markov chains and input-output models, by representing businesses as nodes and the percentage of expenses spent by one business at another as the strength of the edges between the nodes. Markov chain mathematics is used to measure the circulation of currency in the economy by calculating the average number of transactions (average path length) from where the dollar enters the community until it leaves. This method is equivalent to the *multiplier effect* from input-output models but more granular. An example using a simple loop is provided showing different methods of solutions, their equivalence and the impact of loops on the average number of transactions in a small economy.

Keywords: currency, circulation, community, multiplier effect, localization

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1.0 Introduction – Circulation of Currency

Three tools (social network analysis, Markov chains, and input-output models) are examined for their ability to calculate the average number of times a dollar circulates within a community before exiting. None of the three tools can do this calculation single-handedly, but together these tools offer a method for so doing. Research into academic literature has failed to find other examples of using this approach, although some authors appear to come close, without completing the picture. For example, Kichiji & Nishibe (2008) use social network analysis to examine the circulation of community currency, an alternative to bank-backed money, but focus on the distribution of the currency flow, rather than the number of transactions within the community. Other authors (Hoekstra et al., 2007;

Horváth & Frechtling, 1999) attempt to determine currency circulation using input-output models, but do not individualize businesses. Roberts (2005) even focuses on rural economies and decomposes the currency flow using structural path analysis, coming very close to tracking how a dollar circulates in a rural economy; however, this is by sector in an input-output model and not by business. At the risk of overgeneralizing, economic theory literature using Markov chains appears to focus on national levels, and does not examine local economies with granularity (individual businesses).

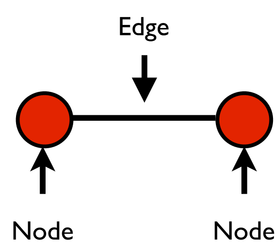
This paper will show how the three tools are equivalent and can be used interchangeably when the data is in the proper form. By using aspects of all three tools, this paper will establish a method to map the circulation of currency in a small economy, and calculate the number of transactions that occur from the time a dollar enters the economy until it leaves. This method is exact and the correct one; all other methods are estimations.

1.1 Social Network Analysis (SNA)

A discussion of the development of and uses for social network analysis is better left to the experts; see Wasserman & Faust (1994), Knoke & Yang (2008), and Hanneman & Riddle (2005) for extensive discussions. Only relevant details will be included here, for highlighting the connections.

SNA data is built from identifying relationships between any two entities, known as “nodes”, within a system, such as people in a community. Connections between nodes can be directed (one way) or undirected (both ways), and are also known as “edges”. The people in a class are the nodes, and if two people have a relationship, there is an edge between the two nodes (see Figure 1.1.1). The connection strength may be a 0 (no connection) or 1 (connection), representing a binary relationship, or may range in values across any arbitrary scale.

Figure 1.1.1: Diagram of Basic SNA Components.



A directed connection infers a difference in position and/or reciprocity. A common example is Alice loves Bob, but Bob loves Carol. The connection is one way. In contrast, a group of individuals playing a team sport are likely to be clustered with undirected connections, indicating a mutually-acknowledged relationship.

Social network data may take one of two forms. The data may be formatted on a line-by-line basis, where each line represents a connection between two nodes. This form has three field values: (a) the originating node, (b) the destination node, and (c) the strength of the connection. This format, known as the “DL” format, is the most common method for storing SNA data for analysis by software (Hanneman & Riddle, 2005).

An alternative form uses a matrix, known as the “adjacency matrix.” It represents individual nodes as rows, with each column in the row representing a connection to the nodes in the system, including itself. This is often denoted in summation notation, and used for the calculation of several metrics relating to positions within the network by individual nodes.

1.2 Markov Chains

Markov chains are based on the 1907 work of A. A. Markov, who studied probability of transitions between multiple states (Grinstead & Snell, 1997, p. 405). These transitions can be sequential, leading to the concept of chains. For example, an object may go from State A to State B to State C, or it may go from State A to State D. The probability of finding the object in State A, B, C, or D at any given point in time is the focus of Markov chain mathematics.

The transitions are usually formatted as a matrix, with each row indicating the probability of transitioning to a different state. If a state cannot be left once arrived at (the probability of transitioning to another state is 0), the Markov chain is known as an "absorbing Markov chain" (Grinstead & Snell, 1997, p. 416). There is a "canonical form" of transition matrices, with the states that can transition to other states at the top of the matrix, and the absorbing states at the bottom. It is not necessary to include any state that an object may exist in prior to entering the system.

In analyzing the absorbing Markov Chain, only the transitional states are examined. The absorbing states are omitted. The "fundamental matrix" is of the form:

1.2.1
$$N = [I - Q]^{-1}$$

where I is the identity matrix and Q is the matrix formed by the transition states, and the exponent of -1 means invert the matrix (Grinstead & Snell, 1997, p. 418). The average number of transitions from any state to an absorbing state and the number of times other states will be entered before reaching an absorbing state can be calculated from the elements in the rows of N. Markov chains have been applied to a wide range of subjects, including queuing theory, ecological food webs, genetics, games, and information theory.

One inspiration for this article is Althoen et al.'s (1993) "How long is a game of Snakes and Ladders?" in which they used Markov chains to calculate the average number of turns for the game in a given layout. That article provided the conceptual foundation for modeling a small economy as a game of Snakes and Ladders (also known as Chutes and Ladders) and using Markov chains. When the economic data forming input-output models are normalized, input-output models are identical to Markov chains.

1.3 Input-output Models

The input-output model was developed in the late 1930s by Wassily Leontief, as a method of calculating the required output necessary by upstream industries to meet input needs of downstream industries as those downstream industries' output changes (Miller & Blair, 2009, p. 1). These demand requirements can be written as linear equations, representing the total demand for a given industry, and these linear equations can be expressed in matrix form. The Leontief inverse represents the "total requirements" of the included industries, recognizing their interdependence:

1.3.1 $$L = (I - A)^{-1}$$

where I is the identity matrix and A is the matrix of the input-output coefficients a_{ij} calculated by the value of the input material i purchased by sector j divided by the value of the industry j (Miller & Blair 2009, p. 21).

One form of a "multiplier effect" can be calculated from the Leontief inverse. This form of the multiplier effect is the direct, indirect and induced increases in economic output necessary to support a given increase in output by a specific industry. If labor is one of these industries, the increased number of jobs can be calculated. Some of these jobs will directly come from the industry, some jobs will be indirectly created from industries that produce products used as raw materials (inputs) for the industry that is increasing its output, and some jobs will be "induced" through increases in these supplier industries.

1.4 Equivalence

In their development of Markov chains, Ching & Ng (2006, p. 3) define the transition matrix P to be the matrix form of $\sum p_{ij}$ where p_{ij} is the probability of transitioning to state j from state i. In contrast, Breuer & Baum (2005) define the probability p_{ij} to be from state i to state j, the inverse of Ching & Ng, but consistent with Bose (2002, p. 149). Breuer & Baum (2005, p. 81) refer to the "system of traffic equations," while Bose (2002, p. 153) refers to "flow balance equations." These equations are the same as those that form the basis of an input-output model, as given by Miller & Blair (2009, p. 19). Bose also explicitly connects these equations to the matrix form that closely resembles the Leontief inverse. Some notable differences between Markov chains and Leontief inverses exist. Leontief inverses do not have a concept of an "absorbing state," nor are the matrices organized in a specific form. Additionally, the matrix of technical coefficients is constructed vertically instead of horizontally as with Markov chains.

However, these methods result in identical calculations, as long as there is recognition as to whether the matrix was constructed horizontally (SNA-style) or vertically (input-output model-style). The relevant data is extracted from either the first row or first column, respectively. As a result, the average path length of a social network from point A to point B is identical to the average path length of a Markov chain and both are the same as the multiplier effect in input-output models (see Appendix A for a short proof). Since the average path length and the Leontief inverse are constructed and calculated in identical ways, and the Leontief inverse is a widely-accepted method for calculating the multiplier effect, the conclusion is that the average path length for a dollar entering a community until it exits is the same as the multiplier effect, on a more granular scale. Furthermore, social network analysis tools can be used to construct the map of the community's economy and the data from the map can be analyzed with Markov chains to determine the average path length (multiplier) for that community.

Degenne & Forsé (1999) demonstrate that social network analysis is based in graph theory. The identification of a correspondence between graph theory and economic modeling is attributed to "Koopmans [(1951)] and Morgenstern [(1954)]" in Asger Olsen (1992, p. 365). Lesne (2006, p. 239) highlights the "deep and operational relation between graph theory and Markov chain theory, the former providing demonstrative and constructive tools to the latter." Graph theory is fundamentally concerned with networks of all kinds, and utilizes matrices for calculations (Wallis, 2007). It is, in fact, graph theory that unifies all three: (a) input-output models, (b) social network analysis, and (c) Markov chains. This allows the tools from one field to be used in analyzing data from the other fields.

2.0 Loops

2.1 Social Network Analysis Focuses on Shortest Paths, not Average Paths

Social network analysis focuses primarily on the shortest path between two nodes, called the geodesic path (Knoke & Yang, 2008). The contribution by loops is not normally considered. A review of the books on social network analysis by Wasserman & Faust (1994), Knoke & Yang (2008), and Friemel (2008) lack any mention of calculating the impact of loops, and only Friemel actually mentions loops at all. While not specifically a book on social network analysis, Newman's (2010) treatment of networks includes applications to social sciences, and contains a thorough discussion of average path lengths in a network with loops (examined as "random walks," a term from Markov chains), as well as several other useful calculations, including the number of possible paths in a network with loops. Consideration of loops within social networks can be seen as a reinforcement of social capital, in the same manner in which economic activity increases in the presence of loops, as will now be shown.

2.2 Mapping and Calculating the Impact of Loops

For illustrative purposes comparing matrices with a geometric series solution, an example of mapping using the smallest possible network that can contain a loop will be used.

Consider two economic chains, with an equal number of businesses (see Figure 2.2.1a and Figure 2.2.1b). The blue circle represents currency entering the community and the red circle represents it leaving. For the chain represented in Figure 2.2.1a, assuming each business spends 100% of its expenses with the next business in the chain, the total economic activity is the sum of the transactions. If each transaction is $100, and each arrow represents a transaction, the total economic activity is $400 (dollars flow in the direction of the arrows, products and services flow in the opposite direction).

Figure 2.2.1a: Five Nodes in a Chain.

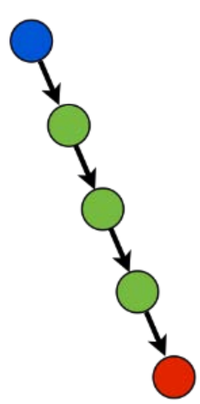
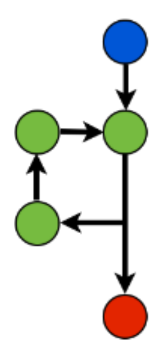
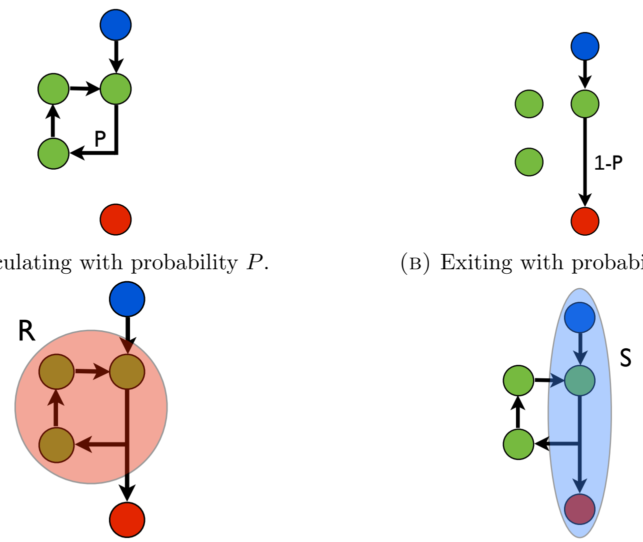


Figure 2.2.1b: Five Nodes with a Loop.



For the chain represented in Figure 2.2.1b, the situation is different. Let P be the probability that the dollar will circulate through the loop (see (A) in Figure 2.2.2), while 1-*P* is the probability the dollar will escape (see (B) in Figure 2.2.2). Let *R* be the length around the loop of green circles (See (C) in Figure 2.2.2, R is equal to 3), and let *S* be the length from the blue circle to the red circle (see (D) in Figure 2.2.2, *S* is equal to 2). The path length *L* is the average number of times a dollar circulates before escaping to the red circle. This is calculated from the weighted probability the dollar escapes. Each dollar has a probability *P* of looping each time, adding the length of the loop to the dollar's path length. A dollar that escapes contributes to the average path length of the economy its path length to that point multiplied by the probability it escapes.

Figure 2.2.2: Components of a Simple Five Node Loop.



(A) Recirculating with probability P. (B) Exiting with probability $1 - P$. (C) Recirculation length R (D) Straight length S

If the percentage of recirculation is 50%, half of the dollars escape during each period of time, but half recirculate. Although the economic impact of this recirculation diminishes as half escapes each loop, the contribution to the economic activity remains for many loops. For example, if a dollar escapes directly with probability 1-*P* (see (B) in Figure 2.2.2), the contribution to the average path length of the system is:

(2.2.1) $$L_0 = (1 - P) \cdot S$$

If the dollar circulates once and then escapes, its average path length is the sum of the length of the loop (*R*) plus the length of the path to escape (*S*) (the total number of nodes it passed through before escaping), times the probability *P* it looped one time, times the probability 1-*P* it escaped after that one loop:

(2.2.2) $$L_1 = (1 - P) \cdot P \cdot (R + S)$$

A dollar has a probability *P* * *P* of circulating for two loops, but escaping with probability 1-*P*. If it does escape after two loops, its average path length is

(2.2.3) $$L_2 = (1 - P) \cdot P \cdot P \cdot (2 \cdot R + S)$$

While the recirculation loop *R* is included twice, for the two loops, the probability *P* * *P* of looping twice shows a decreasing contribution to the total average path length. This continues indefinitely, as there is a non-zero probability of continuing to loop, even after circulating a large number of times. The total path length (denoted $\bar{L}$) is the sum of the path lengths, up to an infinite number of loops.

(2.2.4) $$\bar{L} = L_0 + L_1 + L_2 + L_3 + L_4 + \cdots L_\infty$$

(2.2.5) $$\begin{aligned}\bar{L} = {} & (1-P)\cdot S \\ & + (1-P)\cdot P\cdot(R+S) \\ & + (1-P)\cdot P\cdot P\cdot(2\cdot R+S) \\ & + (1-P)\cdot P\cdot P\cdot P\cdot(3\cdot R+S) \\ & + \cdots \\ & + (1-P)\cdot P^\infty(\infty\cdot R+S)\end{aligned}$$

This can be solved by recognizing Eqn. 2.2.5 is a geometric series of the form:

(2.2.6) $$\bar{L} = (1-P)\cdot[R\cdot\sum_{n=0}^{\infty} nP^n + S\cdot\sum_{n=0}^{\infty} P^n]$$

where *P* is less than 1, and has the converging solution

(2.2.7) $$\bar{L} = (\frac{P}{1-P})\cdot R + S$$

The average path length (the number of times a dollar circulates) in the above five-node network is the ratio of the probability the dollar will recirculate to the probability it escapes times the length of the loop, plus the length of the straight route.

2.3 Matrix Form

The five-node network can be represented as a matrix, identical in form to the matrix form found in social network analysis where the edges are represented in the rows, as opposed to the columns as in input-output matrices (see Section 1.4). Instead of a binary 0 or 1, the strength of the connection between the two nodes is a number between 0 and 1, representing the probability of a dollar going to the downstream node. Exiting the loop is represented as an absorbing state, with a probability of 1 that the state will transition to itself. Each row in the matrix is a business, with the exception of the last one, which is exiting the loop. Each column is also a business, downstream of the business represented in the row. For example, Business X has a probability *P* of spending a dollar locally with Business Y, and a probability 1-*P* of spending a dollar with a business outside of the local area (see Figures 2.2.2a and 2.2.2b).

(2.3.1)
$$A = \begin{array}{c} \\ enter \\ Bus.X \\ Bus.Y \\ Bus.Z \\ exit \end{array} \begin{array}{c} \begin{array}{ccccc} enter & Bus.X & Bus.Y & Bus.Z & exit \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 1-P \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The identity matrix *I* is:

(2.3.2)
$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Constructing the Leontief inverse/fundamental matrix:

(2.3.3)
$$[I-A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 1-P \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -P & 0 & P-1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last column and row are dropped, as they correspond to the absorbing state, which is omitted in the fundamental matrix, and the inverse is taken of the remaining matrix:

(2.3.4)
$$L = [I-A]^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -P & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}^{-1}$$

This has the solution:

(2.3.5)
$$L = \begin{bmatrix} 1 & 1+\frac{P}{1-P} & \frac{P}{1-P} & \frac{P}{1-P} \\ 0 & \frac{P}{1-P} & \frac{P}{1-P} & \frac{P}{1-P} \\ 0 & \frac{1}{1-P} & \frac{1}{1-P} & \frac{1}{1-P} \\ 0 & \frac{1}{1-P} & \frac{P}{1-P} & \frac{1}{1-P} \end{bmatrix}$$

Summing the first row:

(2.3.6)
$$\sum_{j=1}^{4} l_{1j} = 1 + 1 + \frac{P}{1-P} + \frac{P}{1-P} + \frac{P}{1-P} = \frac{P}{1-P} \cdot 3 + 2$$

(2.3.7)
$$\bar{L} = \frac{P}{1-P} \cdot 3 + 2$$

The average path length is identical to that of the geometric solution found earlier, with a loop size *R* of 3 and an exit length *S* of 2. The geometric series approach is unwieldy when economies have many entities, while the inverse matrix approach can be managed with a spreadsheet (see Appendix B).

2.4 Impact of Recirculation

Using the formula calculated above (Eqn. 2.3.7), the impact of recirculating dollars can be quantified. There are three transactions in the loop *R*, and the length *S* is two (from blue to green to red). For a probability of recirculation *P* of .5, the average path length (multiplier) is 5:

(2.4.1) $$\bar{L} = \frac{.5}{1-.5}\cdot 3 + 2 = \frac{.5}{.5}\cdot 3 + 2 = 5$$

Figure 2.4.2: Average path length $\bar{L}$ as a function of *P*

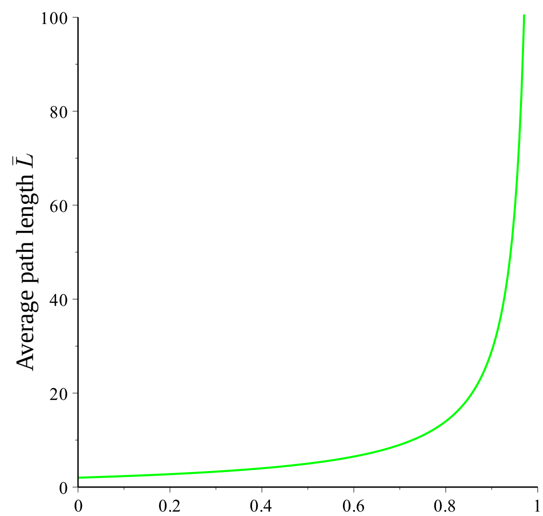


Figure 2.4.2 graphically demonstrates the non-linearly increasing average path length $\bar{L}$ as the probability of recirculation in the three node loop increases. At *P* = 50%, the average path length is 5, while at *P* = 75%, the average path length is 10.

Conversely, Eqn. 2.3.16 can be solved to find the needed inputs to maintain the same output as a function of recirculation, using the multiplier effect in loops to enhance the reduced inputs. Figure 2.2.1b, five nodes with a loop, has a total number of transaction opportunities *R*+*S*, in this case *R*=3, and *S*=2 (refer to Figure (C) and (D) in Figure 2.2.2 for identifying *R* and *S*). If there is no recirculation (*P*=0), there are a total of two transactions. This is the minimum number of transactions that can occur and has a gross economic product of:

(2.4.2) $$T = L \cdot i = 2 \cdot i$$

where *i* is the dollar amount, in this case, \$1. The effective economic inputs needed to maintain this gross economic impact can be reduced by recirculation. The new required economic input can be found by preserving the ratio of the original and new gross economic impact being equal to 1 (again, *i* is equal to 1):

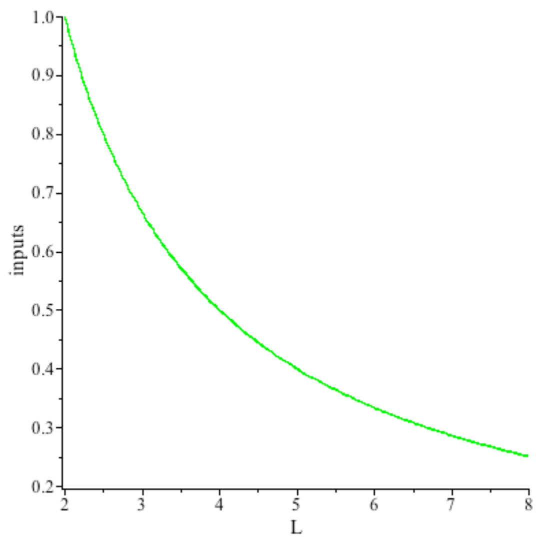
(2.4.3) $$T' = L' \cdot i'$$

(2.4.4) $$\frac{T}{T'} = \frac{2 \cdot i}{L' \cdot i'} = 1$$

(2.4.5) $$i' = \frac{2}{L'}$$

If local Business A has an initial average number of transactions of 2, but agrees to buy 10% of their needed inputs from local Business B who spends 100% of the next transaction out of town, the new average number of transactions for Business A is 2.1 (.9*2+.1*3=2.1, or alternatively, 1 transaction to get to Business A, .90 of the next transaction goes directly out of town while .10 goes one additional transaction, for a total of 1+.9*1+.1*2=2.1). The needed economic inputs to Business A can be reduced to 95.24% (2/2.1) while the system still has the same gross economic impact. This reduction in equivalent economic inputs is graphed in Figure 2.4.3.

Figure 2.4.3: Needed Inputs for Equivalent Outputs Against a Minimum of 2 Transactions.



Note the difference in axis between *L* and *i*. This creates an illusion that there is a maximum point of efficiency in reduction (equivalent to where the slope is greater than -1) somewhere around *L*=4, but in actuality this occurs at the square root of 2, meaning that there are diminishing returns in lowering economic inputs for all average number of transaction values of interest (the minimum being 2).

In a real-world system, the community forest would be a node in the middle of a chain, with goods and services purchased and benefits distributed downstream and access to raw materials sold upstream. That community forest has to balance the potentially higher costs of purchasing locally against the lower revenue of selling locally at a reduced price. Both work to reduce the net income of the community forest, which may impact its ability to distribute grants to local social-responsibility groups. In terms of providing access to raw materials locally, within a value-added chain, the community forest can reasonably look at the chain preceding its position for the multiplier effect and can objectively conclude offering the raw material at half the price while doubling the local economic activity is not being irrational. The downstream path of the expenses also has the same impact; a higher but locally-source quote with added benefits to other local businesses can be a rational choice in the face of a lower but non-local quote. Community forests may find it quite rational to maximize the local collective outcomes through partnerships instead of its own individual outcome in isolation.

2.5 Calculating the Inverse

Microsoft Excel and the Open Office Calc spreadsheet applications can calculate the inverses of matrices. This calculation involves three matrices: the transition matrix, the identity matrix, and the fundamental matrix. Appendix B gives instructions on how to construct these matrices.

Social network analysis software tends to focus on the shortest (geodesic) path length between two nodes, without considering the average path length from one node to another. Inclusion of this metric would be a convenient addition to any SNA package. Many packages include a calculation of the "average path length," but this is the average of the shortest paths between all nodes in the system, which is not the same. For programming environments, "Gauss-Jordan elimination" is a rapid means of finding inverses to matrices. See (Kelly, 2012) for a rigorous treatment and sample code, as well as McMahon (2006) and Sewell (2005) for foundation.

3.0 Additional Details on Case Study of Dunster, British Columbia

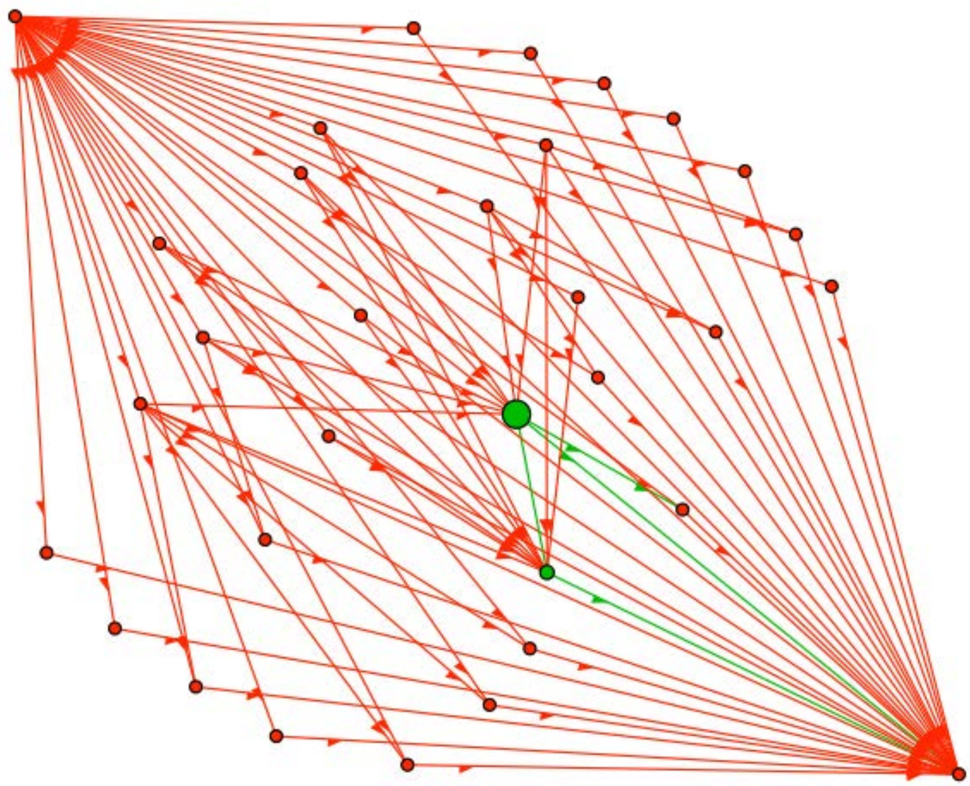
As a "proof of concept," the primary author mapped the economy of the small, rural natural resource-dependent community of Dunster in interior British Columbia, Canada through surveying businesses about where their expenses went. Additionally, the primary author used the data to calculate the average number of times a dollar circulated in the community before exiting the local economy. The full details of the case study are available in the companion article by the authors, "Quantifying Equity with Messrs. Markov, Lorenz and Gini: A Case Study of Dunster, British Columbia," in this issue of JRCD. For brevity in the above article, some details were left out that are now included for completeness.

3.1 Qualitative Analysis: Mapping

Social network analysis contributes two major components to collaboration of these tools: mapping, and centrality. Figure 3.1.1 shows the map of the economy in Dunster.

The arrangement of the network was done manually, with an eye more towards aesthetics than any other attribute. During analysis, the primary author realized that permission to include the business name in the map was not asked in the survey. There is some evidence that a few businesses were participating with an expectation of anonymity. As a result, all publicly available data omits the business name. The circles are individual businesses and the lines represent paths that currency takes, as expenses of the businesses. The circle in the upper left corner represents the dollar entering the economy from outside the community, and the circle in the lower right corner represents the dollar exiting the community economy. The size of the circles was determined by the business' betweenness centrality attribute (Knocke & Yang, 2008). Larger circles mean more currency flows through that business within the community. A visual inspection shows the economy of Dunster to largely be of direct flow, with no identifiable loops. Money appears to pass through Dunster, with a small amount of local expenditures.

Figure 3.1.1: Map of Dunster Economy.



Source: Authors

3.2 Average Number of Transactions: Methodology

As business incomes were not solicited, several models were constructed to provide a range of possible values of each business' income as a percentage of the total expenses of the outside source of income into the community. These models randomized the distribution of incomes to the businesses from outside the local community. This distribution used an algorithm from Weisstein (n.d.) (see Eqn. 3.2.1) to generate a Gaussian (normal) distribution (see Figure 3.2.1a).

(3.2.1) $$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma^2}}$$

The Gaussian distribution is generated from a probability density function, where the probability of the value "x" is calculated based on the mean (μ), the value of x (which ranges from 0 to 1 with intervals equal to 1/30 for this method, there being 30 business in the model), and σ, which is proportional to the spread of the distribution. Larger σ represent a wider, more even distribution of data.

For the modeling, σ values ranging from 0.0205 to 0.035 correspond to a peak value range of 360 and 36 times the least value, respectively. The sum of the values was normalized to 1, so with a σ value of 0.0205 the peak value was 0.072, or 7.2% of the total income into the community, while the least value was 0.0002, or 0.02%. This 360:1 ratio represents a distribution of incomes from $1500 to

$540,000, a range the primary author believes is realistic based on conversations with business owners.

Figure 3.2.1a: Modeled Normal Distribution of Income

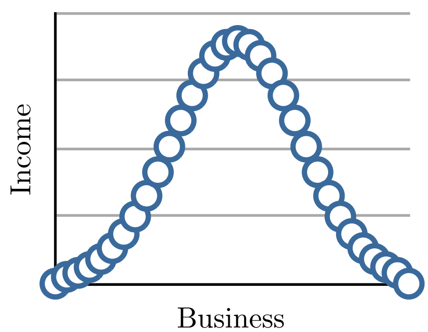
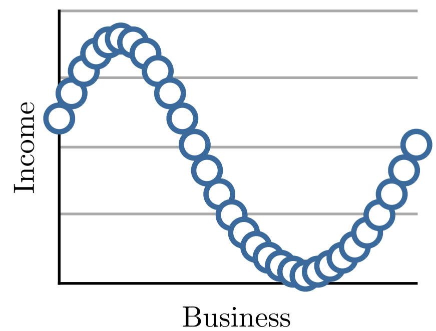


Figure 3.2.1b: A Permutation



However, this may also represent a more narrow concentration of income than may exist in the community. At the lower σ range, the 36:1 ratio represents an income range of $5000 to $180,000, which is likely high at the low end ($5000), and low at the high end ($180,000). However, the distribution is probably more likely near the middle of the spread (i.e., the income distribution is probably not truly "normal"). Given that some of the participating businesses were as small as individuals selling vegetables from their gardens, these are reasonable estimates of the distributions of incomes.

This distribution of shares of currency coming into Dunster was cycled (via permutations, see Figure 3.2.1b) against different arrangements of paths through the community. Each business has an average number of transactions (path length) from that business until the dollar has exited the community. These can be arranged in different models, representing different weightings based on income. The arrangement in Figure 3.2.2a represents the paths being centered (clustered) about the longest average path length by a business in Dunster. Figure 3.2.2b represents a random arrangement, and Figure 3.2.2c represents an organization to minimize the cumulative deviation away from the average number of transactions using a simple mean. This approach reduces instantaneous bias from having a sub-group of businesses with long or short paths. Figure 3.2.2d shows the effects of permutation. (A) in Figure 3.2.2d shows the peak of income aligning with businesses with a low number of transactions, while (B) shows the peak of income aligning slightly past a cluster of business with more transactions.

These steps were necessary to attempt to give a realistic answer and avoid bias. For example, if the businesses with the longest average number of transactions are also given the largest percentage of income into the community, the average path length for the community will be biased towards a higher number than is realistic. Conversations with the business owners in Dunster suggested that there was a significant variation of size of the business and that business' focus on spending locally. Some of the largest businesses spent significantly locally (more than 30% of their expenses), but some also had a high level of expenses with non-local businesses (perhaps 95%). Conversely, some of the smallest businesses spent a lot locally (more than 75%), while others did not (even 0%). Therefore, the most realistic calculation of the average number of transactions is likely to come from the range given by the distribution in 3.2.2c, which is generated through avoiding clustering of long path businesses.

Figure 3.2.2a: Normal Distribution of Path Lengths

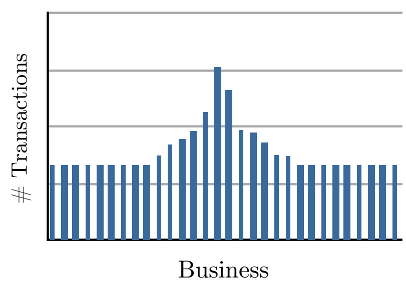


Figure 3.2.2b: Random Distribution of Path Lengths

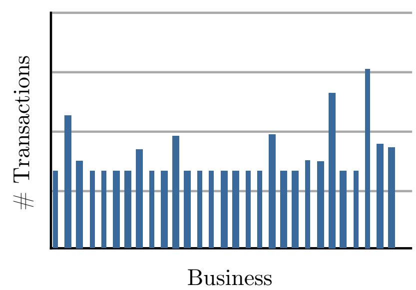


Figure 3.2.2c: Cumulative average

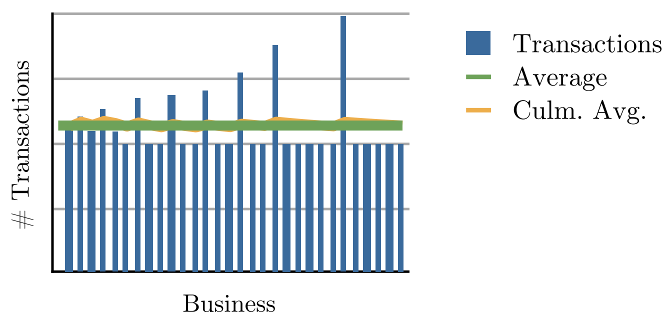
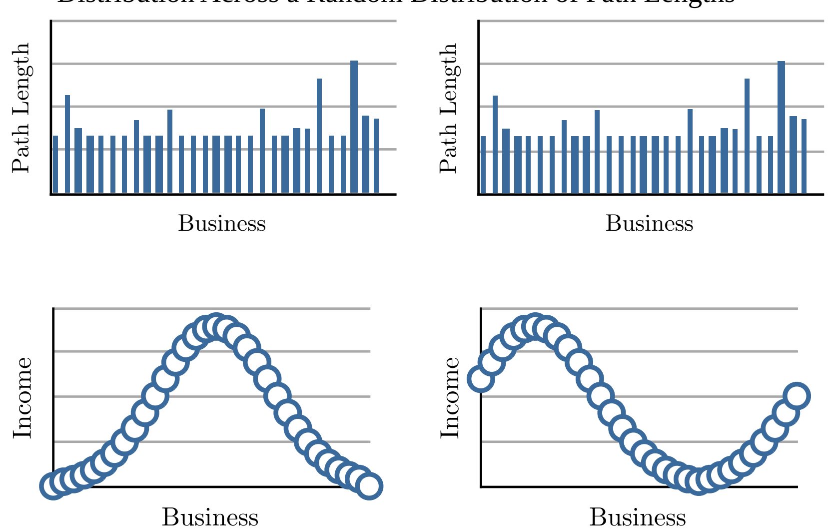


Figure 3.2.2d: Visual Example of Permutation of Income Distribution Across a Random Distribution of Path Lengths



(A) Modeled Normal Distribution of Incomes.

(B) Rotated Permutation of the Normal Distribution of Incomes.

3.3 Quantitative Analysis: Markov Chains

The primary author initially analyzed the data for the Dunster area economy using custom software code to determine the average number of times a dollar circulated within Dunster before leaving the community economy. The primary author wrote the custom software to craft the data into a Markov chain-style matrix and used Gauss-Jordan elimination to determine the average number of transactions that occurred from the time a dollar entered the community until it left the community, as well as the number of transactions for a dollar departing from each business before it left the community. Later, using the methods described in Appendix B, the primary author reproduced the results with spreadsheet software, with only minor differences in calculated values attributable to differences in how floating point operations are done. The data shown below is from the spreadsheet calculations using OpenOffice Calc.

A baseline "Measured" value of the number of times a dollar circulated was derived from the raw data. A second, "Enhanced" model was constructed to show possible opportunities for recirculating local dollars. Not all businesses participated in the survey; for the "Measured" model these non-participating businesses were modeled as having all of their expenses outside the community. The second model, listed as "Enhanced" in Table 3.3.1, modeled 30% of the non-surveyed businesses' expenses being spent locally, as salary to local owners. Personal expenses by residents were not modeled in the Measured model, but in the Enhanced model, 45% of the aggregate resident expenses were modeled as spent locally (an exceedingly unrealistic premise).

Both models were then run against the two different σ values, with the different arrangements as discussed above. The lowest possible number of transactions is 2; the first brings a dollar into a local business, and the second spends that dollar outside the local community.

Table 3.3.1. *Average Number of Transactions in Dunster*

Model	σ **= 0.0205**	σ **= 0.035**
Measured range (average)	2.12 - 2.17 (2.14)	2.13 - 2.16 (2.14)
Enhanced range (average)	2.53 - 2.66 (2.58)	2.54 - 2.64 (2.58)

Table 3.3.1 lists the ranges of average number of transactions for both the Measured and Enhanced models, using the uniform distribution represented in Figure 3.2.2c. Repeated runs of the normally- and randomly-distributed businesses showed bias, as expected, towards the minimum and maximum possible number of transactions. The results from these distributions were not used.

The effect due to income disparity was minimal, suggesting that a simple mean of the number of transactions (plus one, for the incoming transaction) is sufficient to get close to the true number. Additional businesses would make for a wider range of possible values, but beyond a certain point additional businesses are likely to repeat the same paths as other businesses, thereby reinforcing the average.

3.4 Quantitative Analysis: Transactions vs. Centrality and Degree

Table 3.4.2 lists the number of transactions before a dollar spent by a local business exits the community, using "Measured" data, along with many commonly calculated attributes used in social network analysis. For brevity, only the participating businesses and one example non-participating business are included. The betweenness centrality values were used to determine the size of the nodes in the map of the economy in Dunster, while In-Degree was used for the color (see Figure 3.1.1).

Table 3.4.2. *Number of transactions before dollar spent exits community and social network analysis attributes for select businesses in Dunster*

Business #	# Trans-actions	Eccen-tricity	Closeness	Between-ness	In Degree	Out Degree	Degree
30	2.0000	2	1.500	0.000	1	1	2
29	1.7681	2	1.200	0.034	1	4	5
28	1.5547	2	1.250	0.034	1	3	4
26	1.4135	2	1.125	0.034	1	7	8
27	1.3766	2	1.250	0.034	1	3	4
25	1.3547	2	1.200	0.034	1	4	5
24	1.2690	2	1.250	0.034	1	3	4
23	1.2100	1	1.000	0.034	1	2	3
22	1.1345	2	1.333	0.034	1	4	5
21	1.1001	1	1.000	0.034	1	2	3
20	1.0940	1	1.000	8.034	8	3	11
6	1.0000	1	1.000	1.034	11	1	12
8	1.0000	1	1.000	0.034	2	1	3

While the business #20 had highest betweenness, it did not have the longest path/ highest number of transactions of its dollars. In fact, 90% of its dollars exited the community on the very next transaction, due to the need to obtain goods and services from outside the community. While the business was highly respected within the community (as shown by the high number of businesses that engaged in transactions with this business), the business is a "drain" on recirculating currency locally. This highlights an impediment to recirculating dollars: unless local sources of goods and services are present, significant amounts of currency are lost to "importing" those products into the local economy. If the economy is based on obtaining those goods and services at the lowest price, currency may move through rapidly but very little of it will be accumulated into wealth for the community.

The use of betweenness centrality can help identify businesses that are well situated for recirculating currency within the local community, or conversely, are excluded from connection to the local economy. The business with the highest number of transactions obtained all of their income for the business from outside the local community, and had 100% profit paid themselves as salary to a Dunster resident which was then modeled as leaving the community on the next transaction. This inflated their number of transactions, but their betweenness centrality reveals the disconnect from the local economy.

4.0 Conclusions

Through the combination of social network analysis and Markov chains, the economy of a small community can be mapped, and the number of transactions a dollar takes from the time it enters the economy to when it exits can be calculated. As shown in the article, these methods are mathematically identical to the multiplier effect of input-output models, but more granular through the examination of business expenses rather than aggregated sector inputs and outputs. The equivalence of all three means that tools from any of the three can be applied to the data from another.

The case study showed a proof of concept, with maps and calculations. These can be done with open source software for both the mapping (Gephi) and the calculations (OpenOffice). While modeling a range of income distributions is useful to get the breadth of possibilities, using a simple mean is sufficient to be nearly accurate. The case study also highlighted that in rural economies it may be difficult to recirculate dollars within the economy, but that these tools can provide models that give direction on where to focus efforts.

5.0 Appendixes

Appendix A: Proof of Equivalence of Social Network Analysis, Markov Chain Transition and Input-output Matrices

Following the methodology explained in Miller and Blair (2009, p. 244), the Leontief inverse matrix is composed of

(A.1) $$L = (I - A)^{-1}$$

where I is the identity matrix and A is composed of the technical coefficients a_{ij}. These technical coefficients are built from the value of the inputs divided by the total value of the outputs for the sector.

(A.2) $$a_{ij} = \frac{\text{value of inputs from sector } i \text{ bought by sector } j}{\text{total value of the output of sector } j}$$

If z_{ij} is the value of the inputs from sector i bought by sector j and x_j is the total value of the output of sector j, the expression becomes

(A.3) $$a_{ij} = \frac{z_{ij}}{x_j}$$

It is not a requirement that technical coefficient values be normalized, i.e.,

(A.4) $$\frac{\sum_i z_{ij}}{x_j} \neq 1$$

but these values are all less than 1. In an input-output model, goods flow from i to j; by inference, currency flows in reverse, from j to i.

Let s_{ij} be the percent of sector i's expenses that go to each sector j. If the total expenses of sector i are included, including profit as an absorbing value on the diagonal ($i = j$), then the scale of the coefficients a and s are the same.

(A.5) $$s_{ij} = \frac{\text{cost of inputs from sector } j \text{ bought by sector } i}{\text{total costs of sector } i}$$

Social network analysis creates matrices using the strength of the relationship from i to j. These strengths can be normalized.

This difference in direction (flow of goods vs. flow of currency) means that the matrix of technical coefficients A from input-output models is the transpose of the matrix mapping the strength of the relationships in social network analysis, i.e.,

(A.6) $$a_{ij} = s_{ji}$$

(A.7) $$A = S^T$$

The multiplier effect of an increase in demand of a sector in an input-output model is derived from the sum of the elements in the first *column* of the Leontif inverse matrix (Miller & Blair 2007, p. 245). The average path length from the first node in a Markov chain until the exit node is the sum of the elements in the first *row* of the Leontif inverse matrix, although the *S* matrix is minus the row and column associated with the exit node (Althoen et al., 1993). As such, it is necessary to determine if these are, in fact, the same values. A few rules about linear algebra are necessary:

The transpose of the identity matrix is the identity matrix:

(A.8) $$I = I^T$$

The transpose of the sum (or difference) of two matrices is equal to the sum (or difference) of the transposes of the matrices (McMahon 2006, p. 46):

(A.9) $$(M + N)^T = M^T + N^T$$

The transpose of the inverse of a matrix is equal to the inverse of the transpose of the matrix (McMahon 2006, p. 54):

(A.10) $$(M^{-1})^T = (M^T)^{-1}$$

Let *S* represent the social network analysis matrix comprised of the strengths as currency flowing from *i* to *j* as s_{ij} and let *A* represent the input-output matrix comprised of the technical coefficients of goods flowing from *i* to *j* as a_{ij}. From (A. 7):

(A.1) $$A = S^T$$

Construction of the Leontief inverse matrix in input-output model:

(A.12) $$L = (I - A)^{-1}$$

or for Markov chain-based social network analysis:

(A.13) $$L' = (I - S)^{-1}$$

Let

(A.14) $$D = I - S$$

Substituting into (A.13),

(A.15) $$L' = D^{-1}$$

Take the transpose of both sides,

(A.16) $$(L')^T = (D^{-1})^T$$

From (A.10), the transpose of the inverse of *D* is now the inverse of the transpose of *D*,

(A.17) $$(L')^T = (D^T)^{-1}$$

Reversing the substitution from (B.14)

(A.18) $$D^T = (I - S)^T$$

From (A.9)

(A.19) $$(I - S)^T = (I^T - S^T)$$

therefore

(A.20) $$D^T = (I^T - S^T)$$

and from (A.8) and (A.11)

(A.21) $$D^T = (I - A)$$

Substituting into (A.20)

(A.22) $$(D^T)^{-1} = (I - A)^{-1}$$

Taking the inverse of both sides

(A.23) $$(D^T)^{-1} = (I - A)^{-1}$$

Using (A.17)

(A.17) $$(L')^T = (D^T)^{-1}$$

and (A.23)

(A.24) $$(L')^T = (I - A)^{-1}$$

which concludes with using (A.12)

(A.25) $$(L')^T = L$$

The Leontif inverse matrix of social network analysis is the transpose of the Leontief inverse matrix of input-output models. The rows of the Leontif inverse matrix of social network analysis are equal to the columns of the Leontif inverse matrix of input-output models, and therefore, the average path length of a social network analysis-based Markov chain is equal to the multiplier effect of a normalized input-output model.

Appendix B: Spreadsheet Markov Chains

All of the described calculations can be done in a spreadsheet if it can do an inverse matrix operation. To do the Markov chain calculations for the average number of transactions, create five matrices. The first is the result of the survey of businesses. All matrices for this will be an N+2 x N+2 sized matrix, where N is the number of businesses in your community. The first row is the distribution of income to each business. The last row is the node representing the currency leaving the community. Each row in between represents a business. The columns match the rows, i.e., for each business there is a row and a column. The first column is for income coming in, and the last column is for expenses leaving the community. Each column in between is the percentage of expenses spent at the individual community businesses by the business for that row. Expenses leaving the community are entered into the last column, and the last row should have only one entry - a 1 in the last column. The first column should be empty.

As mentioned above, the first row is the distribution of income to each business. This may have to be modeled, unless the businesses are willing to provide their income amounts (unlikely). For the initial pass of analysis, enter 1 divided by the number of businesses for all columns except the first (currency entering the community) and the last (currency exiting the community).

The second matrix is the "Identity" matrix. It is comprised of 1s along the diagonal, i.e., row 3 column 3 has a 1, row 16 column 16 has a 1, etc. All other entries are 0.

The third matrix is [I-A], where I is the identity matrix and A is the business matrix. Each element in A must be subtracted from the corresponding element in the identity matrix. This will create several negative number entries, but as long as the diagonal is positive there should not be problems.

The fourth matrix is the inverse of [I-A], with the exception of the last row and column. Typically this will look like =MINVERSE(Sheet3.A1:Sheet3.AE31), for a 30 business matrix (32 total entries, including the income coming into the community, but the last row and column are the absorbing node, so are dropped from the calculations).

With the successful calculation of the inverse of [I-A], the sum of the each business' row of this matrix is the average number of transactions for that business before its dollars leave the community. Create a column for the sum of each row. Add 1 to get the total number of transactions, including the one that puts the dollar into the business to begin with. These values will be used to calculate the range of values that constitute the minimum and maximum average number of transactions for the community as a whole. This must be done through rotating a distribution of incomes, assuming incomes were not collected during surveying.

To create a normal (Gaussian) distribution as a row, which represents a range of percentages of total community income distributed to each business, use the NORMDIST command once for each column representing a business (not entering and exiting the community). This row will have a 0 in the first and last column, representing 0 probability of money returning the entry node or the exit node. In between are the columns representing a distribution to each business. The NORMDIST command takes four parameters: a number, the mean, the standard deviation, and whether to use the probability density function or cumulative distribution form. Starting at the second column, the number parameter is the fraction of total businesses represented by that column, with the number of the column minus 1 divided by the total number of businesses being the value in the command (COLUMN()-1/N). The mean is 0.5, as the peak of the distribution needs to occur halfway between 0 and 1. The sigma parameter is the width of the income curve. As discussed earlier, the square root of 0.0205 will represent a ratio of 360:1 of highest income to least income. For the last parameter, enter the value appropriate to select probability density function.

This will give a distribution of numbers centered on 0.5. However, the sum of these numbers does not equal 1, which is a necessity for probabilities. Therefore, a second row must be created dividing each entry by the sum of the first row. This second row forms the basis of the permutations that will be used to determine a high and low number of transactions for your community. The permutations cycle the distribution of incomes through all of the businesses, creating conditions where the longest path has the chance to have the highest income, and the shortest path has the lowest income, with variations in between. Since the other paths have lesser or greater percentages of the total income distribution, the calculated values are averaged out.

The first and last column of the permutations will remain 0 at all times. For each permutation, though, the columns will shift to the left, while rolling the left-most business (column 2 if column 1 is the income coming into the community) in that row over to the end on the right (next to the exit column). This can be done through using the OFFSET command, with the first parameter being the current cell, and then -1 for row and +1 for column. This will move the value one row above and one column over into the current cell. Start at the second column of the first permutation, enter the appropriate OFFSET command, and then fill right until one column before the exiting column, then fill down a total number of rows such that the full permutation is equal to the number of businesses N. If you have 30 businesses, you should now have 31 rows - the first non-normalized row (which is not used), then 30 normalized rows with each row shifted one to the left.

Each permutation represents a different distribution of incomes to the local businesses. Rather than recalculate the entire fundamental matrix used to determine the path lengths, each permutation will be multiplied against the final path lengths for the businesses, as a matrix operation. This approach infers that each business' contribution to the total average path length for the community is that business' path length times the percentage of income distributed to that business. For example, a business with an average path length of 2.2 (including the first transaction that brought the dollar to the business) that receives 1% of the total distribution of income contributes 0.022 transactions to the overall community average number of transactions. Rotating these distributions against the fixed number of transactions allows for a more accurate calculation of the average number of transactions, by allowing for a high and a low average number of transactions.

Figure B.1: Permutations and Lengths.

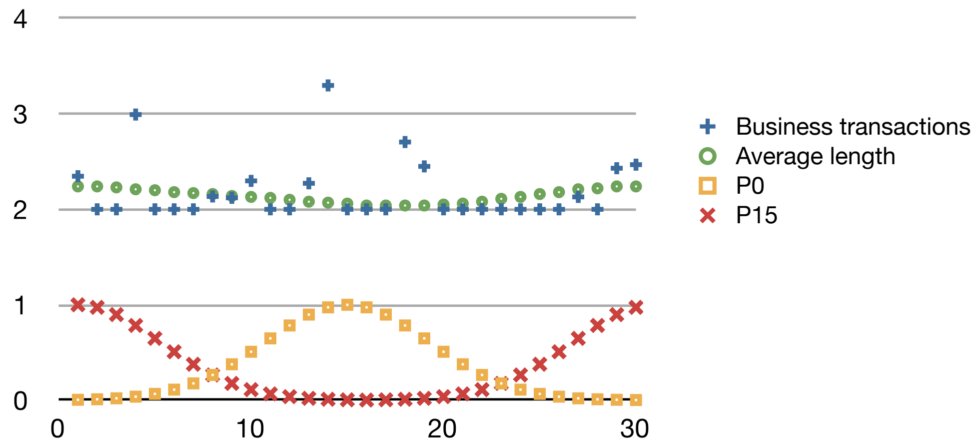


Figure B.1 shows the distribution of business transaction numbers, including the transaction that brings the dollar to the business, ordered in a relatively random way. There appear to be clusters of low numbers of transactions (2) and clusters of businesses with higher numbers of transactions (as high as 3.3). Below that are two permutations of the normal distribution curve used to represent income distribution, scaled to fit on this graph (the peak of 1 is actually 0.091, or 9.1% of the total income). The peak of the income distribution is approximately 15 places to the right of the permutation number. The first permutation, P0, is the original normalized distribution, and it has a peak at approximately 15 on the horizontal axis. The second permutation, P15, is after 15 rotations to the left, and peaks at the 0 and 30 locations on the horizontal axis. P0 lines up well against the middle cluster of businesses that have a higher than average number of transactions, giving them a higher portion of the modeled income distribution. As a result, the average number of transactions for the community is near the peak value (as seen by the high value for the point at 0 on the horizontal axis in the "average length" curve). Conversely, P15 lines up nearly opposite to this, with the least income distribution going to the businesses near the middle cluster (and the value of the "average length" curve is low near 15 on the horizontal axis). As a result, the average number of transactions for this distribution of income to the community is near the lowest value. The curve of the average number of transactions is not a perfect reflection of the normal distribution, as there is some flattening.

If the businesses are arranged in an increasing number of transactions, when the modeled distribution peaks at the same point as the peak of the businesses, the average number of transactions will be artificially increased, as this permutation means the businesses with the longest transaction chains also receives the greatest percentage of the distribution of income from outside the community. Therefore, some randomization is warranted in the absence of specific knowledge of the income distribution from outside the community. As a caveat, though, with enough randomization and permutations, the average number of transactions for the community will tend to range between the least and highest number of transactions by any businesses.

While the "range" is likely to be accurate, a specific single number is often more convenient. Using the average of the permutation results is likely close to the true number. The article discusses altering the order of the businesses so as to not allow clustering to occur. This narrows the range even more, but is difficult to achieve with a spreadsheet. Not using permutations, and just using the even distribution of income used in the initial calculation will lead to a slightly different average number of transactions. For example, for one undiscussed model of the case study published elsewhere in this issue, the randomization of businesses led to a range of 2.04 to 2.24, with an average of 2.14. Using the initial even distribution of income (every business has the same relative income), the calculated path length was 2.19. However, the randomizations are susceptible to bias as a result of the choice of the spread of the distribution. Determining which number is more accurate is at the discretion of the investigator.

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